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THE COMMUTANT OF ANALYTIC TOEPLITZ OPERATORS

BY

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ABSTRACT. In this paper we study the commutant of an analytic Toeplitz operator. For $\phi \in H^\infty$, let $\phi = \chi F$ be its inner-outer factorization. Our main result is that if there exists $\lambda \in \mathbf{C}$ such that χ factors as $\chi = \chi_1\chi_2 \cdots \chi_n$, each χ_i an inner function, and if $F - \lambda$ is divisible by each χ_i, then $\{T_\phi\}' = \{T_\chi\}' \cap \{T_F\}'$. The key step in the proof is Lemma 2, which is a curious result about nilpotent operators. One corollary of our main result is that if $\chi(z) = z^n$, $n \geq 1$, then $\{T_\phi\}' = \{T_\chi\}' \cap \{T_F\}'$, another is that if $\phi \in H^\infty$ is univalent then $\{T_\phi\}' = \{T_z\}'$. We are also able to prove that if the inner factor of ϕ is $\chi(z) = z^n$, $n \geq 1$, then $\{T_\phi\}' = \{T_{z^s}\}'$ where s is a positive integer maximal with respect to the property that z^n and $F(z)$ are both functions of z^s. We conclude by raising six questions.

1. **Introduction.** Let H^2 denote the Hilbert space of functions f analytic in the open unit disc $\mathbf{D}$ for which the functions $f_r(\theta) = f(re^{i\theta})$ are uniformly bounded in L^2-norm for $r < 1$, and let H^∞ denote the linear manifold of bounded functions in H^2. For $\phi \in H^\infty$, T_ϕ (or $T_{\phi(z)}$) is the *analytic Toeplitz operator* on H^2 defined by the relation $(T_\phi f)(z) = \phi(z)f(z)$. These operators have received a great deal of attention recently and many of their properties are well known ([4], [5]). The operator T_z is often called the unilateral shift and is the canonical example of a completely nonunitary isometry of defect one. Every analytic Toeplitz operator commutes with T_z, in fact, every operator that commutes with T_z is an analytic Toeplitz operator. The purpose of this paper is to study the commutant of an arbitrary analytic Toeplitz operator. We obtain some partial results characterizing the commutant of an analytic Toeplitz operator as well as some partial results characterizing those analytic functions whose associated Toeplitz operators have commutant equal to that of T_z. §2 contains our main result stated in terms of pure isometries, §§3 and 4 contain numerous results on the commutant of analytic Toeplitz operators, while §5 contains some open questions.

It is well known that if $f \in H^2$ then there is a function $f^* \in L^2(\mathbf{T})$ such that $f(re^{i\theta})$ converges almost everywhere to $f^*(e^{i\theta})$. $\chi \in H^\infty$ is said to be an *inner*

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function if $|\chi^*(e^{i\theta})| = 1$ almost everywhere (or equivalently, if T_χ is an isometry). Every inner function χ has a factorization $\chi(z) = e^{i\gamma}B(z)S(z)$ with $|e^{i\gamma}| = 1$ where $B(z)$ is a Blaschke product of the form

$$B(z) = z^n \prod_{k=1}^{\infty} \frac{|\alpha_k|}{\alpha_k} \frac{\alpha_k - z}{1 - \bar{\alpha}_k z}, \qquad 0 < |\alpha_k| < 1,$$

and $S(z)$ is a singular inner function of the form

$$S(z) = \exp\left\{- \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t)\right\}$$

with μ a singular measure. $F \in H^\infty$ is said to be an *outer function* if F is of the form

$$F(z) = \exp\left\{\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, k(t)\, dt\right\}$$

where k is a real-valued integrable function (or equivalently, if T_F has dense range). Every nonconstant function $\phi \in H^\infty$ has a unique factorization of the form $\phi(z) = \chi(z)F(z)$ where $\chi \in H^\infty$ is an inner function and $F \in H^\infty$ is an outer function ([6], [10]). Our results will show that this factorization plays a key role in determining the commutant of T_ϕ.

Although we are primarily interested in analytic Toeplitz operators it will be convenient to state some of our results more generally. An isometry V on a Hilbert space $\mathcal{H}$ is called a *pure isometry* ([3], [7]) if $\bigcap_{n=0}^{\infty} V^n\mathcal{H} = \{0\}$. The dimension of the defect space $K_V = \mathcal{H} \ominus V\mathcal{H}$ is called the defect or multiplicity of V, and one easily obtains the decomposition $\mathcal{H} = \Sigma_{n=0}^{\infty} \oplus V^n K_V$. Any two pure isometries of the same multiplicity are unitarily equivalent. Thus V is unitarily equivalent to the unilateral shift U_+ on $l^2_+(K_V) = \Sigma_{n=0}^{\infty} \oplus K_V$ defined by

$$U_+(x_0, x_1, x_2, \cdots) = (0, x_0, x_1, x_2, \cdots).$$

It is easily verified that the commutant of U_+ consists of those bounded linear operators on $l^2_+(K_V)$ of the form $\Sigma_{n=0}^{\infty} \hat{A}_n U_+^n$ where, for A a bounded linear operator on K_V, $\hat{A}$ (the inflation of A) is defined on $l^2_+(K_V)$ by

$$\hat{A}(x_0, x_1, x_2, \cdots) = (Ax_0, Ax_1, Ax_2, \cdots).$$

Thus the commutant of a pure isometry can be characterized.

If $\chi \in H^\infty$ is a nonconstant inner function then T_χ is a pure isometry (this follows since no $0 \neq f \in H^\infty$ is infinitely divisible by χ), which has finite defect if and only if χ is a finite Blaschke product. Hence the commutant of T_χ for χ an inner function is well known, and we attempt to characterize the commutant of an arbitrary analytic Toeplitz operator in terms of these objects. For a Hilbert space $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$ will denote the algebra of all bounded linear operators on $\mathcal{H}$. If $A \in \mathcal{B}(\mathcal{H})$ then $\{A\}'$ will denote the commutant of A, that is, $\{A\}' = \{B \in \mathcal{B}(\mathcal{H}): AB = BA\}$.

2. **Main result.** Throughout this section K will denote a Hilbert space, $\mathcal{H}$ will denote the Hilbert space $l^2_+(K)$, and U_+ will denote the unilateral shift on $\mathcal{H}$.

Lemma 1. *Suppose $S \in \mathcal{B}(\mathcal{H})$ has dense range and commutes with U_+. If $T \in \mathcal{B}(\mathcal{H})$ commutes with SU_+, then T has a lower triangular operator-valued matrix on $\mathcal{H}$.*

Proof. Since T lower triangular is equivalent to T^* upper triangular, which in turn, is equivalent to the subspaces $M_n = \Sigma_{k=0}^n \oplus K$ invariant for T^*, it suffices to prove that T^* leaves M_n invariant. Since T commutes with SU_+, T^* commutes with $S^*U_+^*$ and hence with $S^{*n+1}U_+^{*n+1}$. Thus T^* leaves invariant the null space of $S^{*n+1}U_+^{*n+1}$. Because S has dense range, S^{*n+1} is one-to-one, and

$$\operatorname{null}(S^{*n+1}U_+^{*n+1}) = \operatorname{null}(U_+^{*n+1}) = M_n.$$

Hence T is lower triangular.

The following lemma is essential for our main result.

Lemma 2. *Let N be a nilpotent operator on K and let $X_0 = \lambda I + N$ where $0 \neq \lambda \in \mathbf{C}$. If $B, A_0, A_1, A_2, \cdots \in \mathcal{B}(K)$ satisfy*

(a) $\|A_k\| \le M$, $k = 0, 1, 2, \cdots$, *and*

(b) $A_k X_0 = X_0 A_{k-1} + B$, $k = 1, 2, 3, \cdots$,

then $A_0 = A_1 = A_2 = \cdots$.

Proof. By Theorem 1 in [8], K decomposes into $\Sigma_{i=1}^n \dot{\oplus} K_i$ and X_0 has a lower triangular operator-valued matrix with diagonal elements λI_i. We show that $A_0 = A_1 = A_2 = \cdots$ by showing that they must have the same $(1, n), (1, n-1), \cdots, (1, 1), (2, n), \cdots, (2, 1), \cdots, (n, n), \cdots, (n, 1)$ operator entries with respect to o this decomposition. We repeatedly use the obvious fact that the lemma is true if $N = 0$. More precisely, if $D, C_0, C_1, C_2, \cdots \in \mathcal{B}(K', K'')$ satisfying $\|C_k\| \le M$ and $\lambda C_k = \lambda C_{k-1} + D$ for $k = 1, 2, 3, \cdots$ then $C_0 = C_1 = C_2 = \cdots$. To see that this is the case merely observe that $\lambda C_k = \lambda C_{k-1} + D$ implies $\lambda C_k = \lambda C_0 + kD$. In order that $\|C_k\| \le M$ we must have that $D = 0$ and hence $C_0 = C_1 = C_2 = \cdots$.

Now the $(1, n)$ entry of (b) is

$$\lambda(A_k)_{1,n} = \lambda(A_{k-1})_{1,n} + (B)_{1,n}, \qquad k = 0, 1, 2, \cdots,$$

and so by the above remark and by the fact that (a) implies $\|(A_k)_{i,j}\| \le M$ for $i, j = 1, 2, \cdots, n$, $k = 0, 1, 2, \cdots$ one concludes that $(A_0)_{1,n} = (A_1)_{1,n} = (A_2)_{1,n} = \cdots$. The $(1, n-1)$ entry of (b) is

$$\lambda(A_k)_{1,n-1} + (X_0)_{n,n-1}(A_k)_{1,n} = \lambda(A_{k-1})_{1,n-1} + (B)_{1,n-1}$$

or since $(A_0)_{1,n} = (A_1)_{1,n} = \cdots$, that

$$\lambda(A_k)_{1,n-1} = \lambda(A_{k-1})_{1,n-1} + [(B)_{1,n-1} - (X_0)_{n,n-1}(A_0)_{1,n}]$$

and again we conclude that $(A_0)_{1,n-1} = (A_1)_{1,n-1} = \cdots$.

Now the (i, j) entry of (b) is

$$\lambda(A_k)_{i,j} + (A_k)_{i,j+1}(X_0)_{j+1,j} + \cdots + (A_k)_{i,n}(X_0)_{n,j}$$
$$= \lambda(A_{k-1})_{i,j} + (X_0)_{i,i-1}(A_{k-1})_{i-1,j} + \cdots + (X_0)_{i,1}(A_{k-1})_{1,j} + (B)_{i,j}.$$

Let us now assume that $(A_0)_{p,q} = (A_1)_{p,q} = (A_2)_{p,q} = \cdots$ for all $p = i$, $j < q \le n$ and $1 \le p < i$, $q = j$. Then

$$\lambda(A_k)_{i,j} = \lambda(A_{k-1})_{i,j} + [(X_0)_{i,i-1}(A_0)_{i-1,j} + \cdots + (X_0)_{i,1}(A_0)_{i,j} + (B)_{i,j}$$
$$- (A_0)_{i,j+1}(X_0)_{j+1,j} - \cdots - (A_0)_{i,n}(X_0)_{n,j}]$$

and we conclude that $(A_0)_{i,j} = (A_1)_{i,j} = \cdots$. Inductively we obtain that

$$(A_0)_{i,j} = (A_1)_{i,j} = \cdots \quad \text{for all } i, j = 1, 2, \cdots, n$$

and so $A_0 = A_1 = A_2 = \cdots$.

Corollary 1. *If N is a nilpotent operator on K and $X_0 = \lambda I + N$ where $0 \ne \lambda \in \mathbf{C}$, then*

$$\|X_0^n A X_0^{-n}\| \le M \text{ for } n = 0, 1, 2, \cdots \text{ implies } AX_0 = X_0 A.$$

If K is finite dimensional, then the converse is also true.

Proof. Let $A_k = X_0^k A X_0^{-k}$ for $k = 0, 1, 2, \cdots$. Then $A_k X_0 = X_0 A_{k-1}$ and the result follows from Lemma 2 by setting $B = 0$.

In order to see that the converse is true if K is finite dimensional, first observe that if X_0 satisfies the conclusion of the corollary, that is, $X_0 = \lambda I + N$ for $0 \ne \lambda$ and N nilpotent, then so does any operator SX_0S^{-1} similar to X_0. Now using Jordan canonical forms it is easy to see that for any invertible operator X_0 with two or more distinct eigenvalues there is another operator A satisfying $\|X_0^n A X_0^{-n}\| \le M$ but $AX_0 \ne X_0 A$. For example, if $\mu, \lambda \ne 0$ and if

$X_0 = (\lambda I_1 + N_1) \oplus (\mu I_2 + N_2)$ on $[e_n^{(1)}]_{n=0}^{n_1} \oplus [e_n^{(2)}]_{n=0}^{n_2}$ where $N_i e_n^{(i)}$ equals $e_{n+1}^{(i)}$ if $n < n_i$ and equals 0 if $n = n_i$, if A is defined by $Ae_0^{(2)} = (\lambda/\mu)e_{n_1}^{(1)}$ and 0 otherwise, then $X_0 A X_0^{-1} = (\lambda/\mu)A$. So that if $\lambda \neq \mu$ and $|\mu| \geq |\lambda|$ then $\|X_0^n A X_0^{-n}\| \leq |\lambda/\mu|$ and $AX_0 \neq X_0 A$.

Lemma 3. *Suppose $T \in \mathfrak{B}(\mathcal{H})$ has a lower triangular operator valued matrix on $\mathcal{H}$. If T commutes with $X = (\Sigma_{n=0}^{\infty} \hat{X}_n U_+^n)U_+$ where $X_0 = \lambda I + N$ with $0 \neq \lambda \in \mathbf{C}$ and N nilpotent, then T commutes with U_+.*

Proof. We will show that T commutes with U_+ by inductively proving $T_{k,0} = T_{k+1,1} = T_{k+2,2} = \cdots$ for $k = 0, 1, 2, \cdots$. Notice that $\|T_{k+j,j}\| \leq \|T\|$ for all $k, j = 0, 1, 2, \cdots$.

If $1 \leq j < i$ then the (i, j) entry of $TX = XT$ is

$$\begin{aligned} &T_{i,j+1}X_0 + T_{i,j+2}X_1 + \cdots + T_{i,i}X_{i-j-1} \\ &\qquad = X_{i-j-1}T_{j,j} + X_{i-j}T_{j+1,j} + \cdots + X_0 T_{i-1,j}. \end{aligned} \tag{1}$$

If $i = j + 1$ we obtain $T_{j+1,j+1}X_0 = X_0 T_{j,j}$ and Lemma 2 implies that $T_{0,0} = T_{1,1} = T_{2,2} = \cdots$. Let us now assume that $T_{l,0} = T_{l+1,1} = T_{l+2,2} = \cdots$ for all $l \leq k$. Setting $i = j + k + 2$ in (1) we obtain

$$\begin{aligned} &T_{k+1+j+1,j+1}X_0 \\ &\quad = X_0 T_{k+1+j,j} + [X_1 T_{k,0} + \cdots + X_{k+1}T_{0,0} - T_{k,0}X_1 - \cdots - T_{0,0}X_{k+1}]. \end{aligned}$$

Applying Lemma 2 we obtain that $T_{k+1,0} = T_{k+2,1} = \cdots$ and hence by induction $T_{k,0} = T_{k+1,1} = \cdots$ for all $k = 0, 1, 2, \cdots$. Thus T commutes with U_+

Theorem 1. *Let V be a pure isometry on a Hilbert space $\mathcal{H}$, and $S \in \mathfrak{B}(\mathcal{H})$ have dense range and commute with V. Suppose there exists a $\lambda \in \mathbf{C}$ such that V factors as a product of pure isometries $V_1, V_2, \cdots, V_n$ and such that $S - \lambda I = V_i S_i$ for each $i = 1, 2, \cdots, n$, where each V_i commutes with each S_j. Then $\{SV\}' = \{S\}' \cap \{V\}'$.*

Proof. It clearly suffices to prove the result for $V = U_+$ on $\mathcal{H} = l_+^2(K)$. If T commutes with S and U_+ then it obviously commutes with SU_+. So assume that T commutes with SU_+. Lemma 1 implies that T has a lower triangular operator-valued matrix on $\mathcal{H}$. Since S has dense range and commutes with U_+, it follows that $\lambda \neq 0$ and that $S = \Sigma_{n=0}^{\infty} \hat{X}_n U_+^n$. We need only show that $X_0 - \lambda I$ is nilpotent for then Lemma 3 will imply that T commutes with U_+. Since U_+ is one-to-one and commutes with S, it then follows that T also commutes with S.

We will in fact show that X_0 has a decomposition as described in the proof of Lemma 2. By hypothesis $U_+ = V_1 V_2 \cdots V_n$ and S commutes with each V_i. Since X_0^* is the restriction of S^* to

$$K = \mathcal{H} \ominus U_+\mathcal{H} = (\mathcal{H} \ominus V_1\mathcal{H}) \oplus V_1(\mathcal{H} \ominus V_2\mathcal{H}) \oplus \cdots \oplus V_1 V_2 \cdots V_{n-1}(\mathcal{H} \ominus V_n\mathcal{H}),$$

it follows that X_0^* is upper triangular and hence that X_0 is lower triangular. Let $(X_0)_{ii}$ be the compression of X_0 to $V_1 V_2 \cdots V_{i-1}(\mathcal{H} \ominus V_i\mathcal{H})$. If $f, g \in \mathcal{H} \ominus V_i\mathcal{H}$ then, since $S = \lambda I + V_i S_i$, we obtain

$$\begin{aligned} S(V_1 V_2 \cdots V_{i-1} f) &= (V_1 V_2 \cdots V_{i-1}) S f \\ &= \lambda V_1 V_2 \cdots V_{i-1} f + V_1 V_2 \cdots V_{i-1} V_i S_i f. \end{aligned}$$

But $(V_i S_i V_1 V_2 \cdots V_{i-1} f, V_1 V_2 \cdots V_{i-1} g) = 0$, hence $(X_0)_{ii} = \lambda I_i$.

Remark 1. We remark that in general it is not always true that if V is a pure isometry and if $S \in \mathcal{B}(\mathcal{H})$ has dense range and commutes with V then $\{SV\}' = \{S\}' \cap \{V\}'$. A counterexample is $V = U_+$ on $l_+^2(\mathbf{C}^2)$ and $S = \hat{A}$ where A has eigenvalues ½ and 1.

3. Analytic Toeplitz operators. In this section we reformulate Theorem 1 in terms of analytic Toeplitz operators and obtain numerous consequences.

Theorem 2. *Let $\phi \in H^\infty$ and $\phi = \chi F$ be its inner-outer factorization. If for some $\lambda \in \mathbf{C}$, χ factors as $\chi = \chi_1 \chi_2 \cdots \chi_n$ with each χ_i an inner function and $F - \lambda$ divisible by each χ_i, then $\{T_\phi\}' = \{T_\chi\}' \cap \{T_F\}'$.*

Proof. If χ is constant, then the result is obvious since $\phi = F$. If χ is nonconstant, then as remarked earlier T_χ is a pure isometry and T_F has dense range and commutes with T_χ. By hypothesis there exist $g_i \in H^\infty$ such that $F(z) - \lambda = \chi_i(z) g_i(z)$. Hence $T_F - \lambda I = T_{\chi_i} T_{g_i}$, and of course T_{χ_i} commutes with T_{g_i}. Thus Theorem 1 implies $\{T_\phi\}' = \{T_\chi\}' \cap \{T_F\}'$.

Corollary 2. *Let $\phi \in H^\infty$ and $\phi = \chi F$ be its inner-outer factorization. If*

$$\chi(z) = z^k \prod_{i=1}^{\infty} \left[\frac{|\alpha_i|}{\alpha_i} \frac{\alpha_i - z}{1 - \bar{\alpha}_i z} \right]^{n_i},$$

$\alpha_i \in \mathbf{D}$ distinct, $n_i \le N$, and $F(0) = F(\alpha_i)$ for all i, then $\{T_\phi\}' = \{T_\chi\}' \cap \{T_F\}'$.

Proof. Factor $\chi(z) = \chi_1(z)\chi_2(z) \cdots \chi_N(z)$ where each $\chi_i(z)$ is a Blaschke product in which distinct α_k appear at most once. Since $F(0) = F(\alpha_i)$ for all i, $F - F(0)$ is divisible by each χ_i, and Theorem 2 implies the conclusion.

Corollary 3. *Let $\phi \in H^\infty$ and $\phi = \chi F$ be its inner-outer factorization. If $\chi(z) = ((\alpha - z)/(1 - \bar{\alpha} z))^n$, $n \ge 0$, $\alpha \in \mathbf{D}$ then $\{T_\phi\}' = \{T_\chi\}' \cap \{T_F\}'$.*

Proof. Obvious by Corollary 2.

Proposition 1. *Suppose* $\phi \in H^\infty$ *is such that* $\phi - \phi(\alpha)$ *has a zero of order* $n \geq 1$ *at* $\alpha \in \mathbf{D}$ *and that there exists* $\epsilon > 0$ *such that*

$$|(\phi(z) - \phi(\alpha))/(z - \alpha)^n| \geq \epsilon > 0 \quad \textit{for all } z \in \mathbf{D},\ z \neq \alpha.$$

Then the inner factor of $\phi - \phi(\alpha)$ *is* $((\alpha - z)/(1 - \bar{\alpha}z))^n$.

Proof. The hypotheses imply that $(z - \alpha)^n/(\phi(z) - \phi(\alpha)) \in H^\infty$. Writing

$$\left(\frac{z-\alpha}{1-\bar{\alpha}z}\right)^n = \frac{(z-\alpha)^n}{\phi(z)-\phi(\alpha)}\,(\phi(z)-\phi(\alpha))\left(\frac{1}{1-\bar{\alpha}z}\right)^n$$

and noting that the inner-outer factorization of any function is unique, one easily concludes that the inner factor of $\phi - \phi(\alpha)$ is $((z - \alpha)/(1 - \bar{\alpha}z))^n$ (see p. 51 in [6]).

Remark 2. Since $\{T_\phi\}' = \{T_{\phi-c}\}'$ for any $c \in \mathbf{C}$, Corollary 3 and Proposition 1 enable one to calculate the commutant of a large class of analytic Toeplitz operators. For example, if $\phi(z) = z(\beta - z)$, $0 < |\beta| < 1$, then ϕ does not satisfy the hypothesis of Corollary 2, but some translate of ϕ satisfies Corollary 3 and one can conclude that $\{T_\phi\}' = \{T_z\}'$.

Corollary 4. *If* $\phi \in H^\infty$ *is such that* $\phi - \phi(\alpha)$ *has a simple zero for some* $\alpha \in \mathbf{D}$ *and* $|(\phi(z) - \phi(\alpha))/(z - \alpha)| \geq \epsilon > 0$ *for all* $z \in \mathbf{D}$, $z \neq \alpha$, *then* $\{T_\phi\}' = \{T_z\}'$.

Proof. Proposition 1 implies the inner factor of $\phi - \phi(\alpha)$ is $(\alpha - z)/(1 - \bar{\alpha}z)$. Since $\{T_{(\alpha-z)/(1-\bar{\alpha}z)}\}' = \{T_z\}'$, Corollary 3 implies $\{T_\phi\}' = \{T_z\}'$.

Corollary 5. *If* $\phi \in H^\infty$ *is univalent then* $\{T_\phi\}' = \{T_z\}'$.

Proof. A univalent function satisfies the hypothesis of Corollary 4 for every $\alpha \in \mathbf{D}$.

Remark 3. It is possible to give a direct proof of Corollary 5 without using Theorem 1. Suppose $\phi \in H^\infty$ is univalent and T commutes with T_ϕ. Since $\phi - \phi(\alpha)$ is also univalent for each $\alpha \in \mathbf{D}$, $\phi - \phi(\alpha)$ has inner factor $(\alpha - z)/(1 - \bar{\alpha}z)$. Now T commutes with T_ϕ implies T^* commutes with $T^*_{\phi-\phi(\alpha)}$ for each $\alpha \in \mathbf{D}$. Hence T^* leaves invariant $\operatorname{null}(T^*_{\phi-\phi(\alpha)}) = \operatorname{null}(T^*_{(\alpha-z)/(1-\bar{\alpha}z)}) = \operatorname{span}(K_\alpha)$ where $K_\alpha(z) = (1 - \bar{\alpha}z)^{-1}$. Thus for each $\alpha \in \mathbf{D}$ there exists $\overline{\lambda(\alpha)} \in \mathbf{C}$ such that $T^*K_\alpha = \overline{\lambda(\alpha)}K_\alpha$. Since $|\lambda(z)| \leq \|T^*\| = \|T\|$ for each $z \in \mathbf{D}$ and

$$\begin{aligned}\lambda(z)f(z) = \lambda(z)(f, K_z) &= (f, \overline{\lambda(z)}K_2) = (f, T^*K_z)\\ &= (Tf, K_z) = (Tf)(z) \quad \text{for all } f \in H^2.\end{aligned}$$

Therefore $\lambda \in H^\infty$ and T is the analytic Toeplitz operator T_λ. This method of proof was first used by A. Shields and L. Wallen in [13].

Remark 4. For $\phi \in H^\infty$ define the cluster set of ϕ by

$$C(\phi) = \{\lambda \in \mathbf{C}: \exists z_n \in \mathbf{D},\ |z_n| \to 1,\ \phi(z_n) \to \lambda\}$$

then the hypothesis of Corollary 4 can be rephrased as: If $\phi \in H^\infty$ is such that $\phi - \phi(\alpha)$ has a simple zero for some $\alpha \in \mathbf{D}$ and $\phi(\alpha) \notin C(\phi)$. A significant fact concerning the cluster set of ϕ is contained in the next proposition (although the result is not new [5], the proof is).

Proposition 2. *The Fredholm spectrum of an analytic Toeplitz operator* T_ϕ *is exactly the cluster set* C_ϕ.

Proof. Recall that λ is not in the Fredholm spectrum of an operator T if and only if $T - \lambda$ is Fredholm, that is $T - \lambda$ has closed range and the null spaces of $T - \lambda$ and $(T - \lambda)^*$ are both finite dimensional. For convenience we consider $\lambda = 0$. Note that $T_\phi - \lambda = T_{\phi - \lambda}$ is always one-to-one unless $\phi(z) \equiv \lambda$.

Suppose $0 \notin C(\phi)$ and write $\phi = BSF$ where B is a Blaschke product, S is a singular inner function, and F is an outer function. If S were nonconstant or if B were an infinite Blaschke product, then one could easily find $z_n \in \mathbf{D}$, $|z_n| \to 1$, such that $S(z_n)B(z_n) \to 0$ contradicting $0 \notin C(\phi)$ since $F \in H^\infty$. Hence $S(z) \equiv 1$ and B is a finite Blaschke product. Since $0 \notin \overline{\phi(\mathbf{D})}$ implies $|\phi(e^{i\theta})| = |F(e^{i\theta})| \geq \epsilon > 0$ almost everywhere, it follows that $F^{-1} \in H^\infty$ and that T_F is invertible. But T_B is clearly Fredholm, hence $T_\phi = T_B T_F$ is also Fredholm.

Conversely suppose that T_ϕ is Fredholm and write $\phi = BSF$. Since T_ϕ has closed range, ϕ is bounded below [9], and so T_F is invertible. If S were nonconstant or if B were an infinite Blaschke product then $\dim(\text{null } T_B^* T_S^*) = +\infty$. Since $\text{null } (T_\phi^*) = \text{null } (T_B^* T_S^*)$ we must again have that B is a finite Blaschke product. Thus the inner factor B of ϕ is continuous on $\mathbf{T}$ with $|B(e^{i\theta})| = 1$ and so $0 \notin C(\phi)$.

Remark 5. In the course of proving Proposition 2, we actually show that T_ϕ is Fredholm if and only if χ is a finite Blaschke product and F is invertible in H^∞ where $\phi = \chi F$ is the inner-outer factorization of ϕ. In this case T_χ is Fredholm, and T_χ and T_ϕ both have the same index. The index of T_χ is exactly the negative of the number of zeros of χ counting multiplicity. Furthermore ϕ and χ have the same zeros. If we let ϕ_r be the restriction of ϕ to $C_r = \{z \in \mathbf{C} \mid |z| = r < 1\}$, then the winding number of ϕ_r, $W(\phi_r)$, is the number of zeros of ϕ inside C_r. Since ϕ only has a finite number of zeros, we have $W(\phi_r) = -\text{Index } T_\phi$ for r sufficiently close to 1. Equivalently $\text{Index } T_\phi = -\lim_{r\to 1^-} W(\phi_r)$. If in addition ϕ is continuous on the unit circle, we have $\text{Index } T_\phi = -W(\phi_1)$ where ϕ_1 is the restriction of ϕ to the unit circle. Now in terms of Fredholm operator Corollary 4 can be restated as follows: If T_ϕ is Fredholm operator of index -1, then $\{T_\phi\}' = \{T_z\}'$.

4. **More on $\{T_\phi\}'$.** In this section we will completely characterize the commutant of T_ϕ for any function $\phi \in H^\infty$ whose inner factor is z^n, $n \geq 1$. Of course, analogous results also hold for any function $\phi \in H^\infty$ whose inner factor is $((\alpha - z)/(1 - \alpha z))^n$, $n \geq 1$, $\alpha \in \mathbf{D}$.

Lemma 4. *Suppose T commutes with T_{z^n}, $n \geq 1$, and with T_f where $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots \in H^\infty$. Let $p \geq 1$ be the smallest integer for which $a_p \neq 0$, and let $p = qn + r$ where $0 \leq q$ and $0 \leq r < n$. Then T commutes with T_g where*

$$g(z) = (f(z) - a_0)/z^{qn} = a_p z^r + \cdots.$$

Proof. Since T_f commutes with $U_+ = T_{z^n}$ we have that $T_f = \sum_{k=0}^\infty \hat{A}_k U_+^k$. If $q = 0$ then the result follows. So assume $q \geq 1$. By the definition of p we have that $A_0 = a_0 I$, $A_1 = 0, \cdots, A_{q-1} = 0$. Since T commutes with T_f we obtain

$$T\left(\sum_{k=0}^\infty \hat{A}_k U_+^k\right) = \left(\sum_{k=0}^\infty \hat{A}_k U_+^k\right)T \quad \text{or} \quad T\left(\sum_{k=q}^\infty \hat{A}_k U_+^k\right) = \left(\sum_{k=q}^\infty \hat{A}_k U_+^k\right)T.$$

Hence

$$T\left(\sum_{k=0}^\infty \hat{A}_{k+q} U_+^k\right) = \left(\sum_{k=0}^\infty \hat{A}_{k+q} U_+^k\right)T,$$

since T commutes with U_+, and since U_+ is isometric. But by definition of g we have that $T_g = \sum_{k=0}^\infty \hat{A}_{k+q} U_+^k$.

Lemma 5. *Suppose T commutes with T_{z^n}, $n \geq 1$, and with T_f where $f(z) = a_0 + a_r z^r + \cdots + \in H^\infty$, $a_r \neq 0$, $1 \leq r < n$. If r divides n then T commutes with T_{z^r}.*

Proof. Since T_f commutes with $U_+ = T_{z^n}$, we have that $T_f = \sum_{k=0}^\infty \hat{A}_k U_+^k$. Since r divides n, say $n = qr$, we have that A_0 is unitarily equivalent to

$$\begin{pmatrix} a_0 I & & & & 0 \\ Y & a_0 I & & & \\ & Y & \ddots & & \\ & & \ddots & \ddots & \\ & & & Y & a_0 I \end{pmatrix}$$

on $\Sigma_{l=1}^{q} \oplus \mathbf{C}^r$, where $Y = a_r I + N$ on $\mathbf{C}^r$, $a_r \neq 0$, N nilpotent. Since T commutes with $U_+ = T_{z^n}$ we also have that $T = \Sigma_{k=0}^{\infty} \hat{B}_k U_+^k$. In order that T commute with T_f it is necessary that B_0 commute with A_0. But one easily checks that since $Y = a_r I + N$, $a_r \neq 0$, N nilpotent this implies that B_0 is lower triangular on $\Sigma_{l=1}^{q} \oplus \mathbf{C}^r$, which in turn implies that T is lower triangular with respect to the decomposition of H^2 for $V_+ = T_{z^r}$. Notice that

$$T_f - a_0 I = \left(\sum_{k=0}^{\infty} \hat{X}_k V_+^k \right) V_+$$

and $X_0 = Y = a_r I + N$ with N a nilpotent operator. Lemma 3 now applies and we obtain that T commutes with $V_+ = T_{z^r}$.

Lemma 6. *Suppose T commutes with T_{z^n}, $n \geq 1$, and with T_f where $f(z) = a_0 + a_p z^p + \cdots \in H^\infty$, $a_p \neq 0$. If $p = qn + r$ with $0 \leq q$ and $0 < r < n$ then T commutes with T_{z^s} where $s = \text{g.c.d.}\ (r, n)$.*

Proof. By Lemma 4, T commutes with T_g where

$$g(z) = a_p z^r + \cdots .$$

Now if $s = r$ then r divides n and Lemma 5 implies that T commutes with T_{z^s}. Otherwise there exist integers t, $k \geq 0$ such that $s = tr - kn$. Hence $tr = kn + s$, $0 \leq k$, $0 < s < n$. Since T commutes with T_g, T also commutes with $T_g^t = T_{g^t}$ where $g(z)^t = a_p^t z^{tr} + \cdots$. By Lemma 4, T then commutes with T_h where $h(z) = a_p^t z^s + \cdots$, $0 < s < n$. Since s divides n, Lemma 5 now implies that T commutes with T_{z^s}.

Theorem 3. *If $\phi \in H^\infty$ has inner-outer factorization $\phi = \chi F$ where $\chi(z) = z^n$, $n \geq 1$, then $\{T_\phi\}' = \{T_{z^s}\}'$ where $s \geq 1$ is the positive integer which is maximal with respect to the property that both z^n and $F(z)$ are functions of z^s (equivalently: that ϕ is a function of z^s).*

Proof. Clearly if χ and F are both functions of z^s and if T commutes with T_{z^s} then T commutes with T_χ and T_F, and hence with T_ϕ.

Now suppose that T commutes with T_ϕ. Corollary 3 then implies that T commutes with T_{z^n} and T_F where $F(z) = a_0 + a_1 z + a_2 z^2 + \cdots \in H^\infty$. Let $s \geq 1$ be maximal with respect to the property that both z^n and $F(z)$ are functions of z^s. Denote the sequence of integers $p \geq 1$ such that $a_p \neq 0$ by $p_1 < p_2 < p_3 < \cdots$. Then $s = \text{g.c.d.}\ (n, p_1, p_2, \cdots)$. Let $s_k = \text{g.c.d.}\ (n, p_1, p_2, \cdots, p_k)$. Now suppose $p_1 = q_1 n + r_1$ with $0 \leq q_1$ and $0 \leq r_1 < n$. If $r_1 = 0$ then $s_1 = n$ and T

commutes with $T_{z^{s_1}}$, while if $0 < r_1$ then Lemma 6 implies that T commutes with $T_{z^{s_1}}$ since g.c.d. $(n, r_1) =$ g.c.d. $(n, p_1) = s_1$. In either case T commutes with $T_{z^{s_1}}$. From this one concludes that T also commutes with T_{f_1} where $f_1(z) = f(z) - a_0 - a_{p_1} z^{p_1} = a_{p_2} z^{p_2} + \cdots$. Now suppose $p_2 = q_2 s_1 + r_2$ with $0 \le q_2$ and $0 \le r_2 < s_1$. If $r_2 = 0$ then $s_2 = s_1$ and T commutes with $T_{z^{s_2}}$, while if $0 < r_2$ then Lemma 6 implies that T commutes with $T_{z^{s_2}}$ since g.c.d. $(s_1, r_2) =$ g.c.d. $(s_1, p_2) =$ g.c.d. $(n, p_1, p_2) = s_2$. From this one concludes that T commutes with T_{f_2} where $f_2(z) = f(z) - a_0 - a_{p_1} z^{p_1} - a_{p_2} z^{p_2} = a_{p_3} z^{p_3} + \cdots$. Now suppose $p_3 = q_3 s_2 + r_3$ with $0 \le q_3$ and $0 \le r_3 < s_2$. If $r_3 = 0$ then $s_3 = s_2$ and T commutes with $T_{z^{s_3}}$, while if $0 < r_3$ then Lemma 6 implies that T commutes with $T_{z^{s_3}}$ since g.c.d. $(s_2, r_3) =$ g.c.d. $(s_2, p_3) =$ g.c.d. $(n, p_1, p_2, p_3) = s_3$. Continuing in this manner we obtain that T commutes with $T_{z^{s_k}}$ for every k. Hence T commutes with T_{z^s}.

Corollary 6. *Let* $\phi \in H^\infty$ *and* $\phi = \chi F$ *be its inner-outer factorization with* $\chi(z) = z^n$, $n \ge 1$, *and* $F(z) = a_0 + a_1 z + a_2 z^2 + \cdots$. *If there exists an integer* $p \ge 1$ *such that* $a_p \ne 0$ *and* g.c.d. $(n, p) = 1$ *then* $\{T_\phi\}' = \{T_z\}'$.

Proof. Theorem 3 applies and the hypotheses imply that $s = 1$.

Remark 6. Berkson, Rubel, and Williams [2] define an operator $A \in \mathcal{B}(\mathcal{H})$ to be totally hyponormal if $\{A\}'$ consists entirely of hyponormal operators. It follows that $\{A\}'$ must be abelian. An analytic Toeplitz operator T_ϕ is totally hyponormal if and only if $\{T_\phi\}' = \{T_z\}'$. What analytic functions give rise to totally hyponormal analytic Toeplitz operators? Corollaries 4, 5, and 6 yield many examples of such functions. In particular T_{ψ_n} with $\psi_n(z) = z^n(1 + z)$, $n \ge 1$, has commutant equal to $\{T_z\}'$. It is interesting to note that although the commutant of T_{ψ_n} is $\{T_z\}'$, the weakly closed algebra generated by T_{ψ_n} and I is in fact smaller than $\{T_z\}'$ [12]. It is also interesting to note that $\psi_n, \psi_n^2, \psi_n^3 \cdots$ all give rise to totally hyponormal Toeplitz operators. In fact, using Corollary 6 it is possible to describe a large class of functions $\phi \in H^\infty$ such that T_ϕ and all its powers are totally hyponormal. For a related example see p. 177 of [11].

Corollary 7. *If* $n, k \ge 1$ *are positive integers, then* $\{T_{z^n}\}' \cap \{T_{z^k}\}' = \{T_{z^s}\}'$ *where* $s =$ g.c.d. (n, k).

Proof. Observe that if T commutes with T_{z^n} and T_{z^k} then it also commutes with $T_{z^n(1+z^k)}$. Since $1 + z^k$ is outer, Theorem 3 implies that T also commutes with T_{z^s} where s can be described as in the corollary.

Corollary 8. *If* $n, k \ge 1$ *are positive integers and* $0 < |\alpha| < 1$, *then* $\{T_{z^n}\}' \cap \{T_{((\alpha - z)/(1-\bar{\alpha} z))^k}\}' = \{T_z\}'$.

Proof. If T commutes with T_{z^n} and $T_{((a-z)/(1-\bar{a}z))^k}$ then T also commutes with T_ϕ where $\phi(z) = z^n(1 + ((a - z)/(1 - \bar{a}z))^k)$. Since the coefficient of z in the outer part of ϕ is nonzero, Corollary 6 implies the result.

5. **Concluding remarks.** We would like to conclude this paper by raising some questions which naturally arise in our paper.

Question 1. Suppose $\phi \in H^\infty$ has inner-outer factorization $\phi = \chi F$. Must $\{T_\phi\}' = \{T_\chi\}' \cap \{T_F\}'$?

Question 2. Suppose $\phi \in H^\infty$ is nonconstant. Is the zero operator the only compact operator commuting with T_ϕ?

Question 2 has an affirmative answer in case $\phi(z) = z$ [3]. In fact if ϕ is any inner function the answer to Question 2 is yes. Thus an affirmative answer to Question 1 implies an affirmative answer to Question 2.

Question 3. Suppose $\phi \in H^\infty$. Is $\{T_\phi\}' = \{T_I\}'$ where I is some inner function of which ϕ is a function?

Question 4. Suppose $\phi \in H^\infty$. Does $\{T_\phi\}' \neq \{T_z\}'$ imply that ϕ is a function of an inner function which is not a single Blaschke factor?

An affirmative answer to Question 3 obviously implies an affirmative answer to Question 4. For a related question see E. Nordgren [11].

Question 5. Suppose $\phi \in H^\infty$. If T commutes with T_ϕ does there exist an operator Y on L^2 that commutes with L_ϕ and that leaves H^2 invariant such that $T = Y|_{H^2}$?

Since Question 5 has an affirmative answer in case ϕ is an inner function, an affirmative answer to Question 3 implies an affirmative answer to Question 5. Question 5 asks whether the commutant of the subnormal operator T_ϕ can be lifted to the commutant of the minimal normal extension L_ϕ. Examples are known of subnormal operators (e.g. $0 \oplus T_z$) whose commutants do not lift.

Question 6. Suppose $\{\chi_\alpha\}_{\alpha \in A}$ is a family of inner functions. Is $\bigcap_{\alpha \in A}\{T_{\chi_\alpha}\}' = \{T_I\}'$ where I is some inner function of which each χ_α is a function?

A preprint of J. Ball [1] of the University of Virginia establishes a strong connection between Question 4 and Questions 1 and 6 if the commutant is replaced by the set of projections in the commutant.

In this paper we have given partial answers to the above questions. For example, if the inner factor of $\phi \in H^\infty$ is z^n, $n \geq 1$, then Questions 1 through 5 have affirmative answers. However, in general, the questions seem very difficult and will certainly require techniques different from those presented in this paper.

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